Φ and Music

The evolution of musical scales indicates that musical horizons can expand despite the limitations of the human ear. The correspondence of simple integer ratios with a the perception of musical consonance was noticed early on and it was only much later that this was shown to have a physical basis with respect to the physics of sound and the anatomy of the human ear. The notion of musical dissonance is, however, not as easily defined. A distinction must be made between the ideas of physical/physiological dissonance and conceptual dissonance. The latter, unlike the former, is not necessarily defined in terms of consonance.

Though integer ratios composed of large odd integers[1] or higher primes[2] can be classified as conceptually dissonant, the measure of that dissonance is given in terms of a divergence from consonance that eventually runs runs up against the limitations of the human ear. Dissonance can, however, not only be defined as a lack of consonance but also as a mapping between systems of consonances that takes place within the context of an expanded set of tonal materials that is conceptually based. The twelve keys of equal temperment can be thought of as islands of consonance in an expanded sea of tonal materials through which we can navigate by means of modulations.

For Φ to serve as the medium in which such islands exist it must be capable of generating the integer ratios that make up the basic consonances of music so that it is possible to navigate between them. Proof that the integers can be generated using Φ is given elsewhere, but proof that it is capable of artistic expression of enough breadth to make it more then a mere curiosity is in it's use. There is, however, some indication that the ratios of just intonation are related to Φ through the fibonacci series. Whether this can be dismissed as coincidence or whether Φ may have always played a part in musical aesthetics, through it's relation to the fibonacci series, is difficult to say.

[1] n-odd-limit tuning, Harry Partch, Genesis of a Music: An Account of a Creative Work, Its Roots, and Its Fulfillments, second edition, enlarged (New York: Da Capo Press, 1974), p. 73.

[2] n-prime-limit tuning, Lou Harrison, American composer

The Fibonacci Series and Just Intonation

the fibonacci series:

$$0, 1, 1, 2, 3, 5, 8, 13, \ldots, x_n, (x_n + x_{n-1}) \tag{1}$$

two variations of the above series:

$$1/1, 2/1, 3/2, 5/3, 8/5, \ldots, \frac{x_n}{x_{n-1}}, \frac{x_{n+1}}{x_n} \tag{2}$$

$$1/1, 1/2, 2/3, 3/5, 5/8, \ldots, \frac{x_{n-1}}{x_n}, \frac{x_n}{x_{n+1}} \tag{3}$$

2nd variation brought within the octave:

$$2/1, 1/1, 4/3, 6/5, 5/4, \ldots, \frac{2x_{n-1}}{x_n}, \frac{2x_n}{x_{n+1}} \tag{4}$$

The series represented by series 2 converges to the golden ratio ($\Phi \approx 1.618034$) and the series represented by series 3 converges to it's inverse ($\Phi^{-1}(\phi) \approx .618034$). In the following analysis, an $\uparrow$ will be used to indicate scale degress associated with elements from series 2 and a $\downarrow$ will be used for those from series 4.

Comparing of the first 5 elements of both series, with the degrees of the just major and minor scales shows that all but the 2nd and 7th degree of each scale are accounted for.

Major:

$$1 \updownarrow, 9/8, 5/4 \downarrow, 4/3 \downarrow, 3/2 \uparrow, 5/3 \uparrow, 15/8, 2 \updownarrow \tag{5}$$

Minor:

$$1 \updownarrow, 9/8, 6/5 \downarrow, 4/3 \downarrow, 3/2 \uparrow, 8/5 \uparrow, 16/9, 2 \updownarrow \tag{6}$$

Modulations of the base series can be accomplished by simply multiplying it's elements by a modulating factor. Modulations of the elements of one of the base series by an element from either base series can be thought of as a 1° modulation and modulations of the elements of either base series by a modulating factor taken from a series generated by a 1° modulation can be thought of as a 2° modulation. Though, further modulations are, of course, possible, they are not needed to illustrate the fact that the fibonacci series can be used to generate most of the common ratios associated with various systems of just intonation.

Twelve 12 new ratios not in either base series arise from 1° modulations (see $3-14$ in Table 1) and these in turn give rise to 24 2° modulations, the first 12 of which involve modulating factors that are also important scale degrees of just intonation (see items 15-26, Table 1). An inventory of the scale degrees associated with the 12 1° modulations and the first 12 2° modulations is given in Table 2 where the scale degrees of 9/8, 15/8, and 16/9 are now accounted for by 1° modulations.

A comparison of Table 2 with the tones that occur in justly-intoned scales, as given by Helmholtz[3] in his analysis of comparative dissonance, shows a remarkable correspondence. The only tones not in Helmholtz's table are 25/24, 27/25, 50/27, and 48/25, all of which can be found in the table given by Ellis[4] in his table of Musical Intervals. Furthermore, the only tones in Helmholtz's table that are not in Table 1 are 256/225 and 225/128 and these can be accounted for by modulating 8/5 ↑ and 5/4 ↓ by the 2° modulations of 64/45 and 45/32 from lines 19 and 16 in Table 1.

Further examination of the scales contained in Tables 3, 4, 5, 6, 7 and 8 shows a high degree of correspondence with ratios that can be derived from the fibonacci series. Though the diversity of scales, past and present and from around the world, defies any attempt to explain every variation by means of a single theory, it is worth noting that ratios such as 11/7 and 7/6[5] can be related to the fibonacci series if the three terms (x_{n-1}, x_n, and x_{n+1}) from series 2 and 3 are rearranged as follows:

$\frac{x_{n-1}}{x_n}, \frac{x_{n+1}}{x_{n-1}}$

Then starting with the octave instead of the fundamental results in:

[3]Helmholtz, Hermann von (1885), On the sensations of tone as a physiological basis for the theory of music, Second English Edition, translated by Alexander J. Ellis. London: Longmans, Green, and Co., p.332

[4]Helmholtz, Hermann von (1885), On the sensations of tone as a physiological basis for the theory of music, Second English Edition, translated by Alexander J. Ellis. London: Longmans, Green, and Co., p.453

[5]see scale #3, Table 7 and scale #8, Table 8; respectively

$1/2, 3/1, 4/3, 7/4, 11/7, \ldots, \frac{x_{n-1}}{x_n}, \frac{x_{n+1}}{x_{n-1}}$
from which 11/7 is directly obtainable and from which 7/6 can be derived by the modulation $4/3 \times 7/4 = 7/3 \Rightarrow 7/6$.

So what does it all mean?

The fact that the Fibonacci series can be used to generate many intervals of historical interest as well as the majority of intervals relating to those commonly assigned to the major and minor modes of just-intonation is encouraging whether it is a coincidence or not. Whether this means that Φ has played a part in the aesthetic development and use of tonal systems is unclear but the presence of the Fibonacci Series suggests the possibility that it could do so in the future.

#	factor	type	*series (adjusted)*				
			1	2	3	4	5
1	1/1 ↑	0°	1/1	2/1	3/2	5/3	8/5
2	1/1 ↓	0°	2/1	1/1	4/3	6/5	5/4
3	3/2 ↑	1°	3/2	3/2	9/8	5/4	6/5
4	3/2 ↓	1°	3/2	3/2	2/1	9/5	15/8
5	4/3 ↑	1°	4/3	4/3	2/1	10/9	16/15
6	4/3 ↓	1°	4/3	4/3	16/9	8/5	5/3
7	5/4 ↑	1°	5/4	5/4	15/8	25/24	2/1
8	5/4 ↓	1°	5/4	5/4	5/3	3/2	25/16
9	5/3 ↑	1°	5/3	5/3	5/4	25/18	4/3
10	5/3 ↓	1°	5/3	5/3	10/9	2/1	25/24
11	6/5 ↑	1°	6/5	6/5	9/5	2/1	48/25
12	6/5 ↓	1°	6/5	6/5	8/5	36/25	3/2
13	8/5 ↑	1°	8/5	8/5	6/5	4/3	32/25
14	8/5 ↓	1°	8/5	8/5	16/15	48/25	2/1
15	9/8 ↑	2°	9/8	9/8	27/16	15/8	9/5
16	9/8 ↓	2°	9/8	9/8	3/2	27/20	45/32
17	15/8 ↑	2°	15/8	15/8	45/32	25/16	3/2
18	15/8 ↓	2°	15/8	15/8	5/4	9/8	75/64
19	16/9 ↑	2°	16/9	16/9	4/3	40/27	64/45
20	16/9 ↓	2°	16/9	16/9	32/27	16/15	10/9
21	9/5 ↑	2°	9/5	9/5	27/20	3/2	36/25
22	9/5 ↓	2°	9/5	9/5	6/5	27/25	9/8
23	10/9 ↑	2°	10/9	10/9	5/3	50/27	16/9
24	10/9 ↓	2°	10/9	10/9	40/27	4/3	25/18
25	16/15 ↑	2°	16/15	16/15	8/5	16/9	128/75
26	16/15 ↓	2°	16/15	16/15	64/45	8/5	4/3
27	25/24 ↑	2°	25/24	25/24	25/16	125/72	5/3
28	25/24 ↓	2°	25/24	25/24	25/18	5/4	125/96
29	25/16 ↑	2°	25/16	25/16	75/64	125/96	5/4
30	25/16 ↓	2°	25/16	25/16	25/24	15/8	125/64
31	25/18 ↑	2°	25/18	25/18	25/24	125/108	10/9
32	25/18 ↓	2°	25/18	25/18	50/27	5/3	125/72
33	48/25 ↑	2°	48/25	48/25	36/25	8/5	192/125
34	48/25 ↓	2°	48/25	48/25	32/25	144/125	6/5
35	36/25 ↑	2°	36/25	36/25	27/25	6/5	144/125
36	36/25 ↓	2°	36/25	36/25	48/25	216/125	9/5
37	32/25 ↑	2°	32/25	32/25	48/25	16/15	128/125
38	32/25 ↓	2°	32/25	32/25	128/75	96/125	8/5

Table 1: Fibonacci Modulations

#	ratio	name
1	1/1	Unison
2	25/24	small semitone
3	16/15	just minor 2nd, just semitone
4	27/25	great limma
5	10/9	minor tone, grave 2nd
6	9/8	major tone
7	75/64	augmented tone
8	32/27	Pythagorean minor 3rd
9	6/5	just minor 3rd
10	5/4	just major 3rd
11	32/25	diminished 4th
12	4/3	just 4th
13	27/20	acute 4th
14	25/18	superfluous 4th
15	45/32	false 4th, tritone
16	64/45	diminished 5th
17	36/25	acute diminished 5th
18	40/27	grave 5th
19	3/2	just 5th
20	25/16	grave superfluous 5th
21	8/5	just minor 6th
22	5/3	just major 6th
23	27/16	Pythagorean major 6th
24	128/75	diminished 7th
25	16/9	minor 7th
26	9/5	acute minor 7th
27	50/27	grave major 7th
28	15/8	just major 7th
29	48/25	diminished octave
30	2/1	octave

Table 2: Inventory of Fibonacci tones

Table 3: Pentatonic Scales[a]

#	scale degrees						name
	1	2	3	4	5	6	
1	1/1	9/8	4/3	3/2	5/3	2/1	lacking 3rd and 7th[b]
2	1/1	6/5	4/3	3/2	16/9	2/1	lacking 2nd and 6th[c]
3	1/1	9/8	4/3	3/2	16/9	2/1	lacking 3rd and 6th[d]
4	1/1	9/8	5/4	3/2	5/3	2/1	lacking 4th and 7th[e]

[a]Helmholtz, Hermann von (1885), On the sensations of tone as a physiological basis for the theory of music, Second English Edition, translated by Alexander J. Ellis. London: Longmans, Green, and Co., pp 259-260

[b]ascending

[c]descending; belongs to most Scotch airs with minor character

[d]Gaelic (probably old Bagpipe tune)

[e]characteristic of Scotch airs and Chinese Temple hymns

Table 4: Pythagorean Scales[a]

#	scale degrees								name
	1	2	3	4	5	6	7	8	
1	1/1	9/8	81/64	4/3	3/2	27/16	243/128	2/1	Lydian
2	1/1	9/8	5/4	4/3	3/2	5/3	16/9	2/1	Ionic (Hypophrygian)
3	1/1	9/8	32/27	4/3	3/2	27/16	16/9	2/1	Phrygrian
4	1/1	9/8	32/27	4/3	3/2	128/81	16/9	2/1	Eolic
5	1/1	256/243	32/27	4/3	3/2	128/81	16/9	2/1	Doric
6	1/1	256/243	32/27	4/3	-	128/81	16/9	2/1	Mixolydian[b]
7	1/1	9/8	81/64	729/512	3/2	27/16	243/128	2/1	Syntonolydian

[a]Helmholtz, Hermann von (1885), On the sensations of tone as a physiological basis for the theory of music, Second English Edition, translated by Alexander J. Ellis. London: Longmans, Green, and Co., pp.515

[b]Ellis (in Helmholtz) gives this degree as 588 cents for which there is no obvious corresponding ratio

Table 5: Ancient Greek Scales[a]

#	*scale degrees*								name
	1	2	3	4	5	6	7	8	
1	1/1	10/9	5/4	4/3	3/2	5/3	15/8	2/1	Lydian
2	1/1	9/8	5/4	4/3	3/2	5/3	9/5	2/1	Ionic
3	1/1	10/9	6/5	4/3	3/2	5/3	9/5	2/1	Phrygrian
4	1/1	9/8	32/27	4/3	3/2	128/81	16/9	2/1	Eolic
5	1/1	246/243	32/27	4/3	3/2	5/3	15/8	2/1	Doric
6	1/1	16/15	32/27	4/3	64/45	8/5	16/9	2/1	Mixolydian
7	1/1	9/8	32/27	4/3	3/2	27/16	16/9	2/1	Pythagorean
8	1/1	9/8	5/4	45/32	3/2	5/3	15/8	2/1	Hypolydian

[a]numbers 1-8 from Helmholtz(pg 268,515)

Table 6: Helmholtz's 8 tone scales[a]

#	*scale degrees*								name
	1	2	3	4	5	6	7	8	
1	1/1	9/8	5/4	4/3	3/2	5/3	15/8	2/1	Lydian mode[b]
2	1/1	10/9	5/4	4/3	3/2	5/3	16/9	2/1	Ionic[c]
3	1/1	9/8	6/5	4/3	3/2	5/3	9/5	2/1	Phrygrian mode[d]
4	1/1	9/8	6/5	4/3	3/2	8/5	9/5	2/1	Eolic mode[e]
5	1/1	16/15	6/5	4/3	3/2	8/5	16/9	2/1	Doric[f]
6	1/1	10/9	6/5	4/3	3/2	5/3	16/9	2/1	-[g]
7	1/1	9/8	6/5	4/3	3/2	5/3	15/8	2/1	ascending minor scale

[a]scales 1-7 are from Helmholtz's rational construction of 8 tone scales. Helmholtz, Hermann von (1885), On the sensations of tone, pp. 274-275)

[b]corresponds to the major mode.

[c]also called the Hypophrygian mode, this is Helmholtz's mode of the 4th

[d]this is Helmholtz's mode of the minor 7th.

[e]also called Hypodoric mode, this is a descending minor scale

[f]this is Helmholtz's mode of the minor 6th

[g]this is Helmholtz's mode of the minor 7th

Table 7: Ptolemaic Scales[a]

#	*scale degrees* 1	2	3	4	5	6	7	8	name
1	1/1	16/15	10/9	4/3	3/2	8/5	5/3	2/1	Didymus' chromatic
2	1/1	28/27	10/9	4/3	3/2	14/9	5/3	2/1	Soft Chromatic
3	1/1	22/21	8/7	4/3	3/2	11/7	12/7	2/1	Intense Chromatic
4	1/1	28/27	32/27	4/3	3/2	14/9	16/9	2/1	Middle Soft Diatonic[b]
5	1/1	21/20	7/6	4/3	3/2	63/40	7/4	2/1	Soft Diatonic
6	1/1	9/8	5/4	4/3	3/2	5/3	15/8	2/1	Ptolemaic sequence (ascending)
7	1/1	16/15	6/5	4/3	3/2	8/5	16/9	2/1	Ptolemaic sequence[c]
8	1/1	28/27	16/15	4/3	3/2	14/9	8/5	2/1	Archytas' Diatonic[d]
9	1/1	12/11	6/5	4/3	3/2	18/11	9/5	2/1	Equable Diatonic

[a]versions of Greek scales as given by Ptolemy (in Hawkins, 1:32 as reported by Partch; pp.165,173-174).

[b]attributed to Archytas; of Diatonic genus

[c]agrees with ancient Dorian mode; of Diatonic genus

[d]of enharmonic genus

Table 8: Ancient Greek Tetrachords[a]

#	*scale degrees* 1	2	3	4	name
1	1/1 (3/4)	16/15 (4/5)	-	4/3 (1/1)	Olympos[b]
2	1/1 (3/4)	16/15 (4/5)	10/9 (5/6)	4/3 (1/1)	Old Chromatic[c]
3	1/1 (3/4)	16/15 (4/5)	6/5 (9/10)	4/3 (1/1)	Diatonic
4	1/1 (3/4)	16/15 (4/5)	32/27 (8/9)	4/3 (1/1)	Didymus[d]
5	1/1 (3/4)	256/243 (64/81)	32/27 (8/9)	4/3 (1/1)	Doric[e]
6	1/1 (3/4)	10/9 (5/6)	6/5 (9/10)	4/3 (1/1)	Phrygian
7	1/1 (3/4)	10/9 (5/6)	5/4 (15/16)	4/3 (1/1)	Lydian
8	1/1 (3/4)	21/20	7/6	4/3 (1/1)	Soft Diatonic[f]
9	1/1 (3/4)	12/11	6/5	4/3 (1/1)	Ptolemy's Equal Diatonic[g]
10	1/1 (3/4)	32/31	16/15	4/3 (1/1)	Enharmonic
11	1/1 (3/4)	28/27	16/15	4/3 (1/1)	Ptolemy's Enharmonic
12	1/1 (3/4)	28/27	10/9	4/3 (1/1)	Ptolemy's Chromatic

[a] Tetrachords #1 - #10 are from Helmholtz pp.261,514. #1 - #5, #11 and #12 were given by Ptolemy in his Harmonics

[b] ancient Greek enharmonic of Olympos.

[c] agrees with data of Erastothenes(3rd century B.C.).

[d] the interval of 9/10 in #3 is replaced with one of 8/9 so that when 2 tetrachords are connected the 8/9 is more closely related with 1/1.

[e] constructed via 5ths (Pythagorean)

[f] makes use of interval of the quasi consonance of 6/7, occassionally used by unaccompanied singers. Helmholtz, pg. 264

[g] contains a minor 3rd divided as evenly as possible