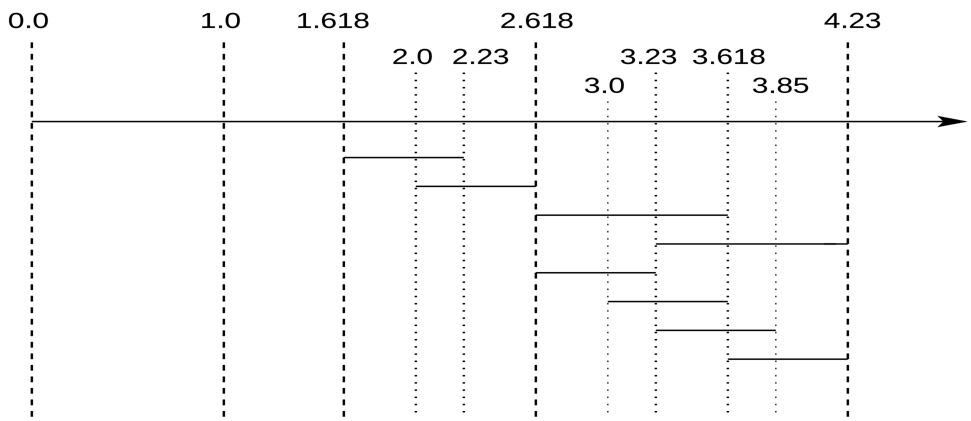

Figure 1: recursively generated segments

Generation of the integers from Φ[1]

Conjecture

The recursive subdivision of Φ, as illustrated by Figure 1, eventually yields all the integers.

Problem:

Given any interval $[a, b], a < b$ on the real axis

the two intervals $[a, (1-\gamma)a + \gamma b]$ and $[\gamma a + (1-\gamma)b, b]$

[1]The above proof was furnished by Chris Gurwood after presenting him with a conjecture based upon my observations. Any errors are most likely due to my transcription of his handwritten notes. I know he did not consider it a paricularly difficult proof, so I assume it is for the most part correct and leave it to others to make that determination.

are generated from it,

where $\gamma = \frac{\sqrt{5}-1}{2}$ is the positive root of the equation $z^2 + z = 1$.

Consider the intervals obtained when this process is applied recursively beginning with any of the intervals

$[\gamma^{n+1}, \gamma^n]$.

Show that every natural number occurs as the endpoint of some such interval.

Solution:

Establish the correspondence $\begin{pmatrix} a \\ \sigma \end{pmatrix} \leftrightarrow [a, a+\delta]$, as 2 vectors.

i.e. represent the intervals with the first component the left hand endpoint and the second component the length of the interval.

Under the first transformation

$[a, b] \to [a.(1-\gamma)a] = [a, a+\gamma(b-a)]$

thus $\begin{pmatrix} a \\ \delta \end{pmatrix} \to \begin{pmatrix} a \\ \gamma\delta \end{pmatrix} = S \begin{pmatrix} a \\ \delta \end{pmatrix}$

where $S = \begin{pmatrix} 1 & 0 \\ 0 & \gamma \end{pmatrix}$

Under the second transformation,

$[a, b] \to [\gamma a + (1-\gamma)b, b] = [a + (1-\gamma)(b-a), a + (b-a)]$,

the new length is again $\gamma(b-a)$.

So $\begin{pmatrix} a \\ \delta \end{pmatrix} \to \begin{pmatrix} a + (1-\gamma)\delta \\ \gamma\delta \end{pmatrix} = T \begin{pmatrix} a \\ \delta \end{pmatrix}$

Where $T = \begin{pmatrix} 1 & 1-\gamma \\ 0 & \gamma \end{pmatrix} = \begin{pmatrix} 1 & \gamma \\ 0 & \gamma \end{pmatrix}$

The left hand endpoints of the intervals are the natural? numbers.

$\begin{pmatrix} 1 & 0 \end{pmatrix} TS^{r_k - 1} TS^{r_{k-1}-1} \ldots TS^{r_2 - 1} TS^{r_1 - 1} \begin{pmatrix} \gamma^n \\ \gamma^{n-1} - \gamma^n \end{pmatrix}$

$$= \begin{pmatrix}\gamma^n & 0\end{pmatrix} TS^{r_k - 1} \dots TS^{r_1 - 1} \begin{pmatrix}1 \\ \gamma\end{pmatrix}$$

where $r_1, \dots, r_k \geq 1$ and $TS^{l-1} = \begin{pmatrix}1 & \gamma^2 \\ 0 & \gamma\end{pmatrix} \begin{pmatrix}1 & 0 \\ 0 & \gamma^{l-1}\end{pmatrix} = \begin{pmatrix}1 & \gamma^{l+1} \\ 0 & \gamma^l\end{pmatrix}$

In general,

$$\begin{pmatrix}1 & ca_k \\ 0 & a_k\end{pmatrix} \begin{pmatrix}1 & ca_{k-1} \\ 0 & a_{k-1}\end{pmatrix} \dots \begin{pmatrix}1 & ca_1 \\ 0 & a_1\end{pmatrix} = \begin{pmatrix}1 & c(a_1 + a_1 a_2 + \cdots + a_1 a_2 \dots a_k) \\ 0 & a_1 \dots a_k\end{pmatrix}$$

Thus

$$\begin{pmatrix}1 & \gamma^{r_k+1} \\ 0 & \gamma^{r_k}\end{pmatrix} \dots \begin{pmatrix}1 & \gamma^{r_1+1} \\ 0 & \gamma^{r_1}\end{pmatrix} = \begin{pmatrix}1 & \gamma(\gamma_1^r + \gamma^{r_1+r_2} + \cdots + \gamma^{r_1+r_2+\cdots+r_k}) \\ 0 & \gamma^{r_1+\cdots+r_k}\end{pmatrix}$$

and

$$\begin{pmatrix}\gamma^n & 0\end{pmatrix} TS^{r_{k-1}} \dots TS^{r_1 - 1} \begin{pmatrix}1 \\ \gamma\end{pmatrix}$$

$$= \begin{pmatrix}\gamma^n & 0\end{pmatrix} \begin{pmatrix}1 & \gamma(\gamma_1^r + \gamma^{r_1+r_2} + \cdots + \gamma^{r_1+r_2+\cdots+r_k}) \\ 0 & \gamma^{r_1+r_2+\cdots+r_k}\end{pmatrix} \begin{pmatrix}1 \\ \gamma\end{pmatrix}$$

$$= \gamma^n [1 + \gamma^2 (\gamma_1^r + \gamma^{r_1+r_2} + \cdots + \gamma^{r_1+r_2+\cdots+r_k})]$$

These numbers are sums of distinct powers of γ.

Conversely, we will show that any sum of distinct powers of γ can be written in this form.

$1 + \gamma^3 + \gamma \dots 0k$

$1 + \gamma^2 + \gamma^{5+} + \dots 0k$

$1 + \gamma + \gamma^{2+} + \cdots + \gamma^{-1}(1 + \gamma^{3+} + \dots)$

Let $Z = \gamma^{a_1} + \gamma^{a_2} + \cdots + \gamma^{a_r}$ with $a_1 < a_2 < \cdots < a_r$.

We must show that Z has a representation of this form
with $a_2 - a_1 \geq 3$.

Assume that among all representations of Z as a sum of distinct powers of r we have chosen one for which r is minimal.
Suppose that for some $1 \leq l < r, a_{l+1} - a_l = 1$.
Let k be the minimal such l.
Then $\gamma^{a_k} + \gamma^{a_{k+1}} = (1 + \gamma)\gamma^{a_k} = \gamma^{-1}\gamma^{a_k} = \gamma^{a_k - 1}$.

Since $a_k - 1$ does not appear as an exponent, we have the representation

$$Z = (\gamma^{a_1} + \cdots + \gamma^{a_{k-1}}) + \gamma^{a_k - 1} + (\gamma^{a_{k+2}} + \cdots + \gamma^{a_r})$$

Where the first sum is 0 if $k = 1$ and the last sum is 0 if $k = r - 1$.

Thus Z is a sum of $r - 1$ distinct powers of γ, contradicting the minimality of r.

So we can assume that $a_{l+1} - a_l \geq 2, 1 \leq l < r$.

Let k be the smallest value of l, $2 \leq l \leq r - 1$, for which $a_{l+1} - a_l > 2$ if such an l exists, and r otherwise.

Then

$$Z = \gamma^{a_1} + (\gamma^{a_2} + \gamma^{a_2+2} + \cdots + \gamma^{a_2+2k-4}) + (\gamma^{a_k+1} + \cdots + \gamma^{a_r})$$

Where the last sum is 0 if $k = r$.

Otherwise we have $a_{k+1} \geq a_2 + 2k - 1$.

$$\gamma^{a_2} + \gamma^{a_2+2} + \cdots + \gamma^{a_2+2k-4} = \gamma^{a_2}(1 + \gamma^2 + \cdots + \gamma^{2k-4})$$

$$= \frac{\gamma^{a_2}(1-\gamma^{2k-2})}{1-\gamma^2}$$

$$= \frac{\gamma^{a_2}(1-\gamma)(1+\gamma+\cdots+\gamma^{2k-3})}{\gamma}$$

$$= \gamma^{a_2+1}(1 + \gamma + \cdots + \gamma^{2k-3})$$

and

$$Z = \gamma^{a_1} + \gamma^{a_2+1} + \cdots + \gamma^{a_2+2k-2} + \gamma^{a_{k+1}} + \cdots + \gamma^{a_r}$$

Where

$(a_2 + 1) - a_1 = (a_2 - a_1) + 1 \geq 2 + 1 = 3$ as required.

So the numbers which occur a left hand endpoints of generated intervals are precisely the numbers which are sums of distinct integral powers of γ. It remains to prove that every natural number has such a representation.

For $n = 1, 2, \ldots$, set $G_n = \gamma^{-n} + (-\gamma)^n$.

Then $G_1 = 1, G_2 = 3$, and in general

$$G_{n+1} = \gamma^{-(n+2)} + (-\gamma)^{n+2}$$

$= \gamma^{-2}\gamma^{-n} + \gamma^{2}(-\gamma)^{n}$

$= (1 + \gamma^{-1})\gamma^{-n} + (1 - \gamma)(-\gamma)^{n}$

$= \gamma^{-n} + \gamma^{-(n+1)} + (-\gamma)^{n} + (-\gamma)^{n+1}$

$= G_n + G_{n-1}$

In particular, the sequence $G_1, G_2 \ldots$ is a strictly increasing unbounded sequence of natural numbers.

For $0 \leq l < k, G_{2k} - G_{2l+1}$ has a representation as a sum of distinct powers of γ, therefore

$G_{2k} - G_{2l+1} = (\gamma^{-2k} + \gamma^{2k}) - (\gamma^{-(2l+1)} - \gamma^{2l+1})$

$= \gamma^{2k} + \gamma^{2l+1} + (\gamma^{-2k} - \gamma^{2l+1})$

$= \gamma^{2k} + \gamma^{2l+1} + (\gamma^{-(2l+2)} - \gamma^{-2l+1}) + (\gamma^{-2k} - \gamma^{-(2l+2)}$

$= \gamma^{2k} + \gamma^{2l+1} + \gamma^{-2l} + (\gamma^{-2k} - \gamma^{-(2l+2)})$

If $l = k - 1, \gamma^{-2k} - \gamma^{-(2l+2)} = 0$.

Otherwise,

$\gamma^{-2k} - \gamma^{-(2l+2)} = \frac{\gamma^{-2k+1} - \gamma^{-2l-1}}{\gamma}$

$= \frac{\gamma^{-2k+1}(1-\gamma^{2(k-l-1)})}{1-\gamma^{2}}$

$= \gamma^{-2k+1}(1 + \gamma^{2} + \cdots + \gamma^{2(k-l-2)})$

$= \gamma^{-(2k-1)} + \gamma^{-(2k-3)} + \cdots + \gamma^{-(2l+3)}$

and

$G_{2k} - G_{2l+1} = \gamma^{2k} + \gamma^{2l+1} + \gamma^{-2l} + (\gamma^{-(2k-1)} + \gamma^{-(2k-3)} + \cdots + \gamma^{-(2l+3)})$

Note that the exponents in this representation lie in the range $[-(2k-1), -2l] \cup [2l+1, 2k]$.

Now let N be an arbitrary natural number and suppose that $N < G_{2n}$. We will show that N has a representation as a sum of distinct powers of γ with all of the exponents in the range $[-(2n-1), 2n]$.

When $n = 1, G_{2n} = 3$ and we have the representations $1 = \gamma^{0}$ and $2 = \gamma^{-1} + \gamma^{2}$.

Assume that for $1 \leq n < k$, every integer $1 \leq N < G_{2n}$ has a representation as a sum of distinct powers of γ with exponents in the range $[-(2n-1), 2n]$.

Let $1 \leq N < G_{2k}$.
We must show that N is a sum of distinct powers of γ with exponents in the range $[-(2k-1), 2k]$.

If $N < G_{2k-2}$. we are done by the induction hypothesis, so we can assume that $G_{2k-2} \leq N < G_{2k}$.

Then $G_{2k} - G_{2k-1} = G_{2k-2} \leq N$.

Let l be the smallest nonnegative integer such that

$G_{2k} - G_{2l+1} \leq N$,

so that $G_{2k} - G_{2l+1} \leq N < G_{2k} - G_{2l-1}$.

Set $N = (G_{2k} - G_{2l+1}) + M$. If $l = 1, M = 0$.

Otherwise,

$0 \leq M < (G_{2k} - G_{2l-1}) - (G_{2k} - G_{2l+1}) = G_{2l+1} - G_{2l-1}$

$= G_{2l}$

By the induction hypothesis, M is either 0 or a sum of distinct powers powers of γ with exponents in the range $[-(2l-1), 2l]$.

$G_{2k} - G_{2l+1}$ is a sum of distinct powers of γ with exponments in the range $[-(2k-1), -2l] \cup [2l+1, 2k]$.

Since these ranges do not intersect, $N = (G_{2k} - G_{2l+1}) + M$ is a sum of distinct powers of *gamma* with exponents in the union of these ranges, namely $[-(2k-1), 2k]$.